literature available thus far no such study has been systematically carried out. The method presented in Ref. 2, it must be pointed out, was meant more for practical three-dimensional lifting surfaces than for airfoils, and to circumvent the above-mentioned problem in calculating generalized airloads in the complex frequency domain.

The method of Ref. 1 for flutter control needs to be refined further so that the relationship between measurements and states can be clearly established in the same fashion as Ref. 2. This is essential if one wishes to construct a reduced-order observer to estimate the states and implement the controller. To see the relationship between measurements and states physically it seems better to work with the system of Eqs. (1), reduce to a set of uncoupled second-order equations, and work back to Eq. (2) rather than reducing them to first order, although the latter is essential for actual computation of the control law. One does not need to work back to Eq. (2) if the relationship between measurements and states is not required as in the case of Ref. 1.

This relationship is especially important for the design of a minimum-order observer (with arbitrary observer dynamics), moreso than eliminating the additional states introduced by the aerodynamic model, as the control law would be a linear functional of only the observer states and measurements. The author's experience has been that such a control law can easily be constructed for a tapered large-aspect-ratio wing with a trailing edge control surface using modified strip analyses, a second-order approximation for Theodorsen's function, and a first-order observer without any measurements of the large number of additional states introduced by the aerodynamic model. A substantial reduction in the required number of measurements was possible when the observer order was increased to two and three. The use of lower order observers seem to be more appropriate than using crude approximations for the aerodynamic model. It is the author's view that the additional states introduced by the aerodynamic models would not only *not* have to be estimated, but would actually make the observer design insensitive to modelling errors – a factor which is certainly important in active flutter control.

Acknowledgments

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References

[1] Edwards, J.W., Breakwell, J.V., and Bryson, A.E. Jr., "Active Flutter Control Using Generalized Unsteady Aerodynamic Theory," *Journal of Guidance and Control,* Vol. 1, Jan.-Feb. 1978, p. 32-40.

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Reply by Authors to R. Vepa

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THE application of Laplace transform techniques to aeroelastic vehicle analysis has been studied in Refs. 1-3, where the validity of unsteady aerodynamic theories for arbitrary values of s was demonstrated with examples drawn from two-dimensional incompressible and supersonic flow. In addition, Refs. 1 and 2 presented an active aeroelastic synthesis technique, termed the Rational Model, and gave examples of active flutter control utilizing this technique. Vepa's comments are directed at 1) aeroelastic analysis techniques for arbitrary complex values of s, and 2) the Rational Model synthesis technique.

Aeroelastic systems are modeled as

$$[M_s s^2 + B_s s + K_s - Q(s)]X(s) = G(s)u(s) \tag{1}$$

where M_s, B_s, and K_s are mass, damping, and stiffness matrices, X is an n-dimensional state vector, $G(s)$ is the control distribution matrix, and $Q(s)$ is the unsteady aerodynamic transfer function matrix relating structural motion to generalized forces. The m-dimensional control input vector u is to be interpreted as a position command to aerodynamic control surface servos.

Aeroelastic analysis involves the determination of the roots of Eq. (1) as a function of Mach number and altitude. To this end, the representation of $Q(s)$ may be in any convenient form. Reference 1 demonstrated that the exact complex roots may be determined if $Q(s)$ is known for general values of s and also gave examples of the use of rational function approximations of $Q(s)$ for determining the roots. Such approximations, which Vepa[4] favors for aeroelastic analysis, require augmenting the state of Eq. (1) in order to model the approximations. It is important to understand that $Q(s)$ generally contains nonrational components, such as Bessel functions, which cannot be represented as rational functions (ratios of polynomials in s). Hence, augmented-state models using rational function approximations of $Q(s)$ cannot be exact models, but examples given in Ref. 3 indicate that they may be very adequate for engineering design purposes.

Vepa expresses concern over the validity of unsteady aerodynamic theory for general values of s in subsonic flow and three-dimensional flow. Edwards[5] has addressed this issue and has given examples of calculations of generalized forces in both of these cases. At issue is the applicability of Laplace transform techniques to the governing partial differential equations. It is possible[5] to cast the integral equation solutions into the form of convolution integrals, from which Laplace transformation on the time variable leads directly to the desired generalized aerodynamic solution. A point of confusion in this derivation is the appearance of integrals which are convergent only for $\mathrm{Re}(s) > 0$. This causes no difficulty since the integrals are only representations of the analytic functions describing the physical solution (e.g., Bessel functions) which are valid throughout the s-plane. It is precisely this fact that accounts for the usefulness of rational function approximations of $Q(s)$ and allows their use for unrestricted values of s.

Reference 5 also resolved a point of confusion regarding the calculation of subsonic indicial response functions using the Laplace inversion integral, which may have precipitated some of Vepa's comments.[1] The characteristic starting pulse contained in such functions was shown to be caused by the Kutta wave generated at the trailing edge by the impulsive motion.

Vepa's comments regarding the difficulty of performing analysis and synthesis calculations for arbitrary values of s seem premature since this effort has yet to be attempted in a realistic design case. It seems unfair to compare a new computational technique with long-used techniques which utilize efficient approximations of complex functions.

Whereas aeroelastic analysis may be viewed as the determination of the "open-loop" roots or poles, aeroelastic synthesis is the process of moving these roots to desirable regions of the s-plane. The complexity and robustness of aeroelastic controllers is obviously directly related to the

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model used in the design synthesis. In Ref. 1 a synthesis technique termed the Rational Model was introduced in which it was shown that the rational portion of the aeroelastic system response, $X(t)$, could be exactly modeled without state augmentation as

$$\begin{bmatrix} \dot{X} \\ \ddot{X} \end{bmatrix} = \left[\begin{array}{c|c} 0 & I \\ \hline f_{21} & f_{22} \end{array}\right] \begin{bmatrix} X \\ \dot{X} \end{bmatrix} + \left[\begin{array}{c} g_1 \\ \hline g_2 \end{array}\right] u \tag{2}$$

An algorithm was given for the construction of F and G which involved the determination of the $2n$ discrete roots of Eq. (1) and appropriate residues evaluated at these root locations. Vepa comments that this implies that $Q(s)$ may be approximated as

$$Q(s) = P_0 s^2 + P_1 s + P_2 \tag{3}$$

for suitable choices of the P_i matrices. His comment that $P_0 = 0$ when $M \neq 0$ appears to derive from assumptions contained in the analysis of Ref. 4 rather than that of Ref. 1 or 3 since the Rational Models have the structure of Eq. (2) regardless of Mach number. The interesting aspect of this point of view is that Eq. (3) (with $P_0 = 0$), when substituted into Eq. (1), may be interpreted as a full-state feedback control law. It can be shown that the $2n$ poles and selected output eigenvectors with dimension equal to the dimension of Q may be arbitrarily assigned by suitable choices of P_1 and P_2. Thus, the exact pole locations and certain output eigenvectors of the aeroelastic system may be matched by the model of Eq. (2) at any given flight condition. This yields a Rational Model that is slightly different from the one described in Ref. (1), one in which the upper submatrix of the control distribution matrix, g_1, can be left as zero.

Finally, Vepa expresses a preference for full-state (or) minimum-order observers for aeroelastic system synthesis. These are alternative synthesis techniques and we would welcome the publication of the study to which Vepa refers.

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